Worcester Polytechnic Institute August, 2014

Linear Algebra General Comprehensive Exam

Print Name: ____________________________ Sign: ____________________________

In the following, $\mathbb{R}^{\mathrm{n}}$ is real n-dimensional space, $\mathbb{C}^{\mathrm{n}}$ is complex n-dimensional space, and $\mathbb{R}^{\mathrm{n}\times\mathrm{n}}$ is the space of real $n \times n$ matrices.

1. Let $A = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 2 & 0 & 0 \\ -1 & 1 & 2 & 1 \\ -1 & 1 & 0 & 3 \end{pmatrix}$.

   a. What is the rank of A?

   b. What is the determinant of A?

   c. Find the eigenvalues and eigenvectors of A.

   d. Find the characteristic polynomial of A.

   e. Find the transformation matrix M and its inverse such that $J = M^{-1}AM$ is the Jordan canonical form of A.

2. Compute the orthogonal complement of the subspace of $\mathbb{C}^4$ spanned by the vectors $(-1, i, 0, 1)$ and $(i, 0, 2, 0)$.

3. a. Determine a basis of $\mathcal{V}_1 \equiv \{p(x) = \alpha x^2 + \beta x^3 : \ \alpha, \beta \in \mathbb{R}\}$ that is orthonormal with respect to the inner product $\langle p_1, p_2 \rangle = \int_{-1}^{1} p_1(x) p_2(x)\, dx$.

   b. What is the orthogonal projection of $p(x) = x^4$ onto $\mathcal{V}_1$ with respect to this inner product.

   c. Define $\mathcal{V}_2 \equiv \{p(x) = \alpha + \beta x : \ \alpha, \beta \in \mathbb{R}\}$ and let $L : \mathcal{V}_1 \to \mathcal{V}_2$ be the linear transformation defined by $L(p) = d^2p/dx^2$ for $p \in \mathcal{V}_1$. What is the matrix representation of L with respect to the basis of $\mathcal{V}_1$ that you found in part (a) and the basis $\{1, x\}$ of $\mathcal{V}_2$? What are the null-space and range of L? Is L invertible?

4. Consider the linear system $Ax = b$, where $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \\ 1 & -1 \end{pmatrix}$.

   a. Discuss the existence and uniqueness of solutions of this system. (Does a solution exist for every b? If not, then for what b, if any, does a solution exist? If a solution exists for some b, is it unique? If not, describe all possible solutions.)

   b. Find a least-squares solution for $b = \begin{pmatrix} 1 & 0 & 1 & 0 \end{pmatrix}^T$ by forming and solving the *normal equation* $A^T Ax = A^T b$.

5. Suppose that $u \in \mathbb{R}^{\mathrm{n}}$ is given. Define the *annihilators* of u to be $\mathcal{A}(u) \equiv \{M \in \mathbb{R}^{\mathrm{n}\times\mathrm{n}} : \ Mu = 0\}$.

a. Show that $\mathcal{A}(u)$ is a subspace of $\mathbb{R}^{\mathrm{n}\times\mathrm{n}}$.

b. Define a linear transformation $P : \mathbb{R}^{\mathrm{n}\times\mathrm{n}} \to \mathbb{R}^{\mathrm{n}\times\mathrm{n}}$ by $P(M) = M(I - uu^T)$ for $M \in \mathbb{R}^{\mathrm{n}\times\mathrm{n}}$. Show that P is a projection onto $\mathcal{A}(u)$.

c. Suppose that $\langle \cdot, \cdot \rangle$ is the Frobenius inner product on $\mathbb{R}^{\mathrm{n}\times\mathrm{n}}$, i.e., $\langle M, N \rangle = \operatorname{trace} MN^T$ for M and N in $\mathbb{R}^{\mathrm{n}\times\mathrm{n}}$. Show that P is *orthogonal* projection with respect to this inner product.

6. Let $\mathcal{F}$ be the vector space of all continuous functions on the interval $[0, 1]$ with the inner product and norm defined by

$$\langle f, g \rangle = \int_0^1 f(t)g(t)\ dt \quad \text{and} \quad \|f\| = \langle f, f \rangle^{1/2}.$$

a. Let $\mathcal{P}$ be the subspace of all polynomials in $\mathcal{F}$.

  i. What is $\dim(\mathcal{P})$?

  ii. Does there exist an orthonormal basis for $\mathcal{P}$?

b. By the Weierstrass Approximation Theorem, any function $f \in \mathcal{F}$ can be uniformly approximated by a polynomial $p \in \mathcal{P}$, i.e., if $f \in \mathcal{F}$ and $\epsilon > 0$, then there exists a $p \in \mathcal{P}$ such that $|f(t) - p(t)| < \epsilon$ for all $t \in [0, 1]$. Suppose that $\|f\| \neq 0$ and that $0 < \delta < 1$. Taking $\epsilon = \delta\|f\|$, we have in particular that there exists a $p \in \mathcal{P}$ such that $|f(t) - p(t)| < \delta\|f\|$ for all $t \in [0, 1]$.

  i. Check whether the following computation is correct:

$$\begin{aligned}\langle f, p \rangle &= \int_0^1 f(t)[f(t) - f(t) - p(t)]\ dt \geq \int_0^1 f^2(t)\ dt - \int_0^1 |f(t)||f(t) - p(t)|\ dt \\ &\geq \|f\|^2 - \|f\|\,\|f - p\| \geq \|f\|^2\,(1 - \delta) > 0.\end{aligned}$$

  ii. What can you conclude concerning $\mathcal{P}^{\perp}$? Compute $(\mathcal{P}^{\perp})^{\perp}$.